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METHOD OF ANALYSIS OF MATHEMATICAL MODELS OF MEDIA UNDER COMPLEX LOADING

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A method of analysis of constitutive models of media with complex loading trajectories is proposed. It is based on a comparison of data from laboratory experiments and corresponding results of numerical calculations. In previous laboratory experiments, nearly homogeneous complex loading with continuous rotation of the principal axes of the strain tensor and loading with broken strain trajectories was performed. Numerical calculations for the types of loading corresponding to the experiments performed are based on the constitutive equations of the model. The numerical results obtained and data of the laboratory experiments are compared. The hypoplastic model of a geomedium is analyzed. Analysis shows that the model is a satisfactory qualitative and quantitative approximation of data from laboratory experiments on complex loading of geomaterials.

Introduction. In solving most problems of the mechanics of continuous media, it is necessary to select or develop a corresponding mathematical deformation model. For an elastic body or a linearly viscous liquid, the conventional classical Lamé equations and Navier–Stokes equations were developed. However, for more complex media, for example, elastoplastic and granular materials and nonlinear fluids, such equations are not available. Therefore, for each class of problems, the choice of a deformation model should be made separately. Here much depends on particular loading conditions and the objectives to be pursued by solution of a particular problem.

Most of the models in use are of a phenomenological nature. This means that they are developed relying on some basic experiments. Such experiments should determine both the equations of the model and the values of the associated parameters.

The choice of basic experiments is a difficult problem. Basically, mathematical models can be developed using any experiments, for example, experiments on forcing of a stamp in a material. However, to interpret such an experiment, one must choose a certain mathematical model, solve a boundary-value problem, compare the results obtained with experimental data, adjust the model, etc. It is a complicated and cumbersome way. At the same time, there is a special class of loadings for which preliminary information on the mathematical model is not required to interpret experiments. This is the class of quasistatic loadings with uniform stress and strain distributions over space. Here the deformation process reduces to a sequence of affine transformations. In this case, only some general restrictions should be satisfied: loading should be such that inertial and mass forces can be ignored, and the deformation process should be stable (i.e., rheological and other forms of instability are not admitted). Thus, in stable processes, the kinematics of deformation does not depend on the rheology of the material, i.e., it is identical for any media: elastoplastic, viscous, free-flowing, etc. Therefore, experiments on homogeneous deformation are basically an ideal basis for development and analysis of mathematical models of continuous media.

It is not practical to achieve a completely homogeneous state. It is only possible to implement a deformation process that is close to homogeneous deformation. For metals, such processes are well known

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[1, 2]. These are tension and twisting thin-walled tubular specimens. However, this classical procedure is unacceptable for more complex media: viscous and granular materials, soils, powders, etc. Here, a search for new basic experiments is required.

A general classification of homogeneous deformation processes including well-known processes such as homogeneous tension, pure shear, torsion, etc., is given in [3, 4]. New classes of loadings have also been detected. They have been used to develop procedures and laboratory setups intended for homogeneous deformation of a free-flowing medium under pure shear [5, 6], complex loading with continuous rotation of the principal axes of the stress tensor [7, 8], and complex loading with broken strain trajectory [9]. In the last case, the principal axes of stresses rotate about the material by a jump through a finite angle. In [10, 11], the hypoplastic model of a free-flowing medium of [12–14] was analyzed using the experimental procedures developed. For this model, numerical experiments on simple loading (pure shear) and complex loading (with continuous rotation of the principal axes of the stress tensor and broken loading trajectory) have been performed. A comparison of results of the numerical calculations and the corresponding data of laboratory experiments indicate that the model of [12–14] gives a good approximation of the experimental results on dilatancy and the stress level and predicts a number of effects involved in the deformation of a free-flowing medium for both simple and complex loading trajectories. At present, a new version of the hypoplastic model of [12–14] is proposed in [15].

The goal of the present work is to perform a numerical analysis of the model of [15] using the experimental procedure of [7–9] for complex loading of a free-flowing medium with continuous rotation of the principal axes of the stress tensor and broken strain trajectories.

Loading Procedure. We first consider homogeneous deformation of a free-flowing medium under complex loading with continuous rotation of the principal axes of the stress tensor. According to the solutions of [7, 8], the specimen of the material must be shaped like an elliptical cylinder with the axis Ox_3, and the velocity vector $\boldsymbol{v}$ satisfying the Kepler's law must be specified on the elliptic boundary in the plane Ox_1x_2 (plane deformation):

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \qquad |\boldsymbol{v} \times \boldsymbol{r}| = \Omega = \text{const}. \tag{1}$$

Here $\boldsymbol{n}$ is the normal vector to the elliptic boundary of the sample and $\boldsymbol{r}$ is the radius-vector (Fig. 1). In this case, irrespective of whether the medium is elastic, viscous, plastic, etc. (if there is stability, i.e., shear localization, fracture, etc. are absent), the strain distribution over space is uniform [7]. Both conditions (1) are difficult to satisfy in practice. It is easier to retain only the basic features of the ideal situation, i.e., to meet the first condition (1) and assume a constant linear velocity:

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \qquad |\boldsymbol{v}| = \text{const}. \tag{2}$$

It is clear that replacement of conditions (1) by (2) leads to a certain inhomogeneity. However, by virtue of symmetry, the material element at the center of the ellipse (Fig. 1) is under complex loading conditions with rotation of the principal axes of stresses. Thus, if measurements are conducted for the central element on a small basis, the results obtained can be considered accurate and can be compared with theoretical calculations for homogeneous deformation.

In [8], loading with boundary conditions (2) was performed as follows. The free-flowing material specimen 4 (Fig. 2) was placed in a cylindrical cup 3 made from a thin bronze plate. The bottom of the cup was closed by stretched rubber 8. The loading device consisted of rigid plates 5 with coaxial elliptic holes in which the cylindrical cup 3 was inserted. Plates 5 enclose the cup at different sections across the height and were fixed to the pin 7 which was supported in bearings on the base 6. Loading was performed by continuous rotation of the pin 7 (and, hence, the plates 5) by an electric motor. The rotating moment on the cylindrical cup that arose as a result of friction against the plates was compensated by flexible ties 2, which were fastened to the upper part of the cup and immovable posts 1 (Fig. 2).

The goal of the experiments of [8] was to study the behavior of the dilatancy of a free-flowing medium, the stress state, and the coaxiality of the stress and strain tensors. During loading, the cross-sectional area of the cup containing the sample of the medium remained constant. This reduces examination of dilatancy to examination of the change in the height of the material filling the cup. The stress state and, hence, the

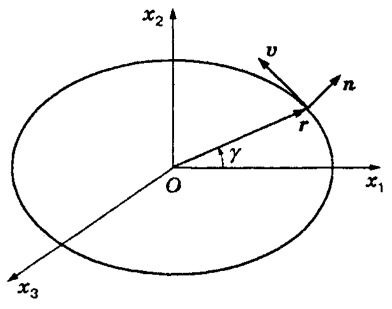

Fig. 1

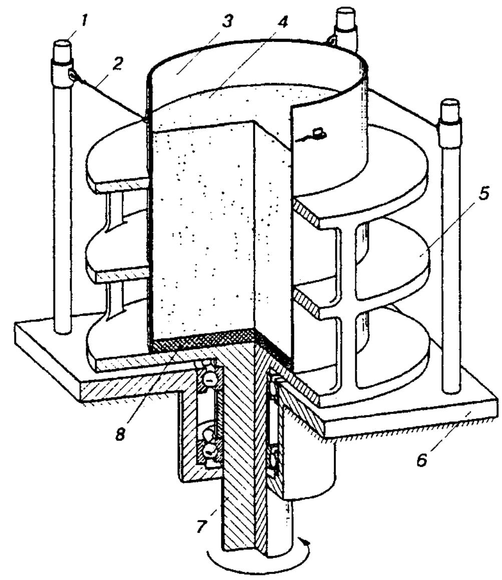

Fig. 2

coaxiality of the stress and strain tensors were studied using a special stress gauge [16], which was placed inside the sample at a certain fixed depth. A vertical spoke was tightly fastened to the gauge to control the orientation of the gauge with respect to the axes of the elliptic holes of the loading plates. A flag whose plane coincided with the plane of the gauge was fixed at the free end of the spoke above the surface of the sample.

We consider the experimental procedure of [9] for complex loading of a free-flowing medium with broken strain trajectory. Loading was performed using a homogeneous shear device [5, 6] (Fig. 3), which was a cubic chamber with rigid movable walls 2, 3, and 6 and rigid square bottom with changing shape. During shear, the square becomes a parallelogram (Fig. 3a and c). This is a typical example of simple loading.

Bending of the loading trajectory can be implemented as follows [9]. The flexible cylindrical cup 1 (Fig. 3b) with the free-flowing material sample 5 was placed in the shear chamber. The lateral surface of the cup was an elastic shell made of thin sheet bronze. The bottom of the cup was closed by uniformly stretched rubber, which was fixed to the inner surface of the shell. The cup was placed at the center of the chamber inside the circular cylindrical cavity formed by a set of vertical plates 4 of different length (Fig. 3b). Upon displacement through angle γ, the lateral walls 3 and 6 of the chamber exerted forces through plates 4 on the lateral surfaces of the cup 1 and deformed it: the circle became an ellipse (Fig. 3a and c). Bending of the loading trajectory was performed as follows. A certain initial position of the chamber, for example, position A was selected (Fig. 3a). Suppose it is necessary to shift the material from position A to position C (Fig. 3c) with the broken trajectory. For this, the chamber was first shifted from position A to position B (Fig. 3b), and then loading was terminated. In this position, the cup 1 containing the material was rigidly rotated through a certain angle æ. In other words, the packing of the particles of the material in the cup was not disrupted. Then, further loading was performed by shift from position B to position C. The magnitude of bending of the loading trajectory is determined by the angle æ. In this case, the loading trajectory in the strain space is a broken two-link line.

The dilatancy of the bulk material and the coaxiality of the stress and strain tensors was examined experimentally. Theoretically, in pure shear, the cross-sectional area of the cup containing the material should remain constant [the area of the circle (Fig. 3b) should be equal to the areas of the ellipses (Fig. 3a and c)]. In practice, the perimeter of the cross section of the cup is kept constant. In this case, the change in the cross-sectional area is a quantity of higher-order smallness and can be ignored. It should be noted that for large shears, it is not difficult to introduce corrections for the change in the cross-sectional area. In other

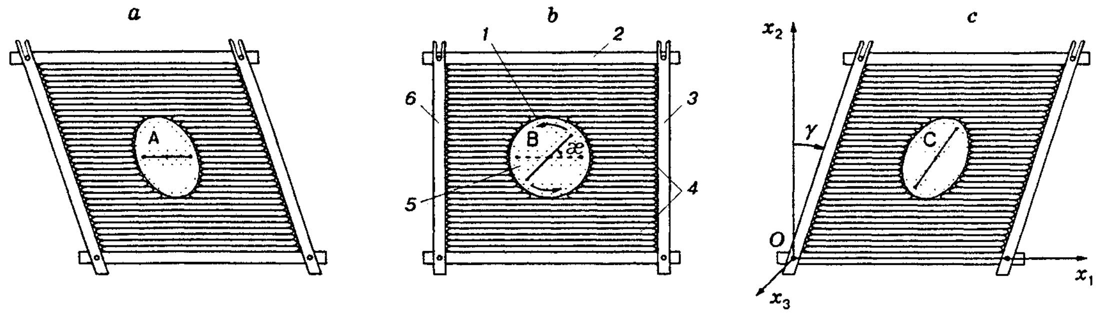

Fig. 3

words, as in the previous experiment, examination of dilatancy reduces to examining the change in the height of the fill. The coaxiality of the stress and strain tensors was checked by a gauge of tangential stresses [17], which was placed at a particular depth in the free-flowing medium. The gauge was used to determine the orientation of domains of the principal stresses (i.e., domains in which shearing stresses were absent).

Numerical Simulation. We perform numerical calculations of the complex loading of a free-flowing medium using the mathematical model of [15]. The constitutive equations of the model relating the stress tensor T and the strain-rate tensor D have the form

$$T^0 = C_1 \operatorname{tr}(T+S)D + C_2 \frac{\operatorname{tr}((T+S)D)}{\operatorname{tr}(T+S)}(T+S) + \Big[\frac{C_3T^2 + C_4T^{*2}}{\operatorname{tr}(T)} + \frac{C_5T^3 + C_6T^{*3}}{\operatorname{tr}(T^2)}\Big]\sqrt{\operatorname{tr}(D^2)},$$

$$\dot{e} = (1+e)\operatorname{tr}(D); \tag{3}$$

$$S = sE, \qquad s = \Big[s_0 + k\Big(\frac{p}{p_0}\Big)^{\nu} \ln\Big(\frac{1+e}{1+e_0}\Big)\Big]\Big(\frac{p}{p_0}\Big)^{\alpha},$$

$$p = \operatorname{tr}(T), \qquad k = -s_0 \Big/ \Big[\Big(\frac{p}{p_0}\Big)^{\nu} \ln\Big(\frac{1+e_r}{1+e_0}\Big)\Big], \tag{4}$$

$$T^0 = \dot{T} - WT + TW, \qquad T^* = T - \frac{1}{3}pE, \qquad W = \frac{1}{2}(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm t}),$$

where $\boldsymbol{v}$ is the velocity field, e is the porosity, and E is the unit tensor. The constants of the model are

$$C_1 = -103.01, \quad C_2 = -197.61, \quad C_3 = 37.24, \quad C_4 = 1572.92, \quad C_5 = -394.69,$$

$$C_6 = -1265.66, \quad p_r = -0.5 \text{ MPa}, \quad p_0 = -0.729 \text{ MPa}, \quad s_0 = -0.149 \text{ MPa}, \tag{5}$$

$$\nu = 0.1, \quad \alpha = 0.6, \quad e_r = 0.73, \quad e_0 = 0.54.$$

Within the framework of the same equations, model (3)–(5) describes both the state of active loading and unloading. By virtue of this, the equations of the model are substantially nonlinear even for increments.

Simulation of complex loading with continuous rotation of the principal axes of stresses was performed as follows. As noted above, boundary conditions (1) guarantee a nonuniform strain distribution. In this case, the homogeneous velocity field in the plane Ox_1x_2 has the form

$$v_1 = \Omega\,\frac{x_2}{b^2}, \qquad v_2 = -\Omega\,\frac{x_1}{a^2}, \tag{6}$$

where a and b are the semiaxes of the ellipse ($a > b$), and motion along the boundary is performed counterclockwise for $\Omega > 0$. We fix a horizontal layer of the material at depth h. The vertical pressure component for the layer can be considered constant and equal to the weight of the overlying layers [10], i.e.,

$$\dot{\sigma}_{33} = 0 \quad \text{or} \quad \sigma_{33} = \rho h = \text{const}, \tag{7}$$

where ρ is the density of the material. By virtue of the homogeneity of deformation for the selected layer, the relations

$$\frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = 0 \tag{8}$$

hold. The tensors D and W take the form

$$D = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & d \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & -q & 0 \\ q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $p = ((a^2 - b^2)/(2a^2b^2))\Omega$ and $q = ((a^2 + b^2)/(2a^2b^2))\Omega$. The quantity $d = \dot{\varepsilon}_{33} = \partial v_3/\partial x_3$ is in fact the required dilatancy parameter. Thus, the problem amounts to solving Eqs. (3)–(5) subject to conditions (6)–(8). In a component-wise form, they are a system of ordinary first-order differential equations with nonlinear right side. This system is solved numerically by the Euler method.

Simulation of complex loading with broken strain trajectory was performed by a similar scheme. If shear loading occurs in the plane Ox_1x_2, the associated homogeneous velocity field has the form

$$v_1 = 2\delta x_2, \quad v_2 = 0, \quad \delta = \pm k_0 = \text{const}, \quad k_0 > 0. \tag{9}$$

Here the plus and minus denote the direction of shear. If the trajectory is bent [rigid rotation of the material sample in position B (Fig. 3b)] through angle æ, the field (9) becomes the field

$$v_1 = 2\delta(-x_1 \sin æ + x_2 \cos æ) \cos æ, \qquad v_2 = 2\delta(-x_1 \sin æ + x_2 \cos æ) \sin æ. \tag{10}$$

Thus, the velocity field (10) is the plane velocity distribution in homogeneous deformation with the loading trajectory bent through angle æ. If æ $= 0$, the field (10) coincides with (9). In this case, as in simulation of complex loading with continuous rotation of the principal stress axes, we select a layer of the material at a certain depth. For this layer, relations (7) and (8) remain valid. In turn, the tensors D and W take the form

$$D = \begin{pmatrix} -\delta \sin 2æ & \delta \cos 2æ & 0 \\ \delta \cos 2æ & \delta \sin 2æ & 0 \\ 0 & 0 & d \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & \delta & 0 \\ -\delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $d = \dot{\varepsilon}_{33} = \partial v_3/\partial x_3$ is a dilatancy parameter. Thus, the problem reduces to solving systems (3)–(5) and (10) subject to conditions (7) and (8).

Calculation Results. We conduct numerical calculations for complex loading of a free-flowing medium with continuous rotation of the principal axes of stresses. For the initial state of the medium, we set

$$\sigma_{33} = 0.003\,75 \text{ MPa}, \quad \sigma_{11} = \sigma_{22} = \xi\sigma_{33}, \quad \xi = 0.42, \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0, \quad e = 0.85. \tag{11}$$

Loading is implemented by the following scheme. We assign a certain value of the areal velocity Ω (for numerical calculations, this value determines the value of the integration step in time and may not be small) and set the ratio of the semiaxes of the ellipse equal to $a/b = 1.1$. We fix a certain point on the elliptic boundary, and, as the loading parameter, we select the angle γ through which this mass point rotates about the center of the ellipse during deformation (see Fig. 1). For the initial state, we set $\gamma = 0$ and increase γ to a certain fixed value, for example, $\gamma = 10\pi$.

In this scheme, as in the experiments, the material undergoes dilatation upon deformation (it is compacted in our case). With time, this process stabilizes, and a steady regime is established. In this regime, the parameters do not change during further deformation. A plot of the strain ε_{33} versus the angle γ is shown in Fig. 4. At the points O and A, the porosity is $e = 0.85$ and 0.81681, respectively. The stresses σ_{11}, σ_{22}, and σ_{12} depending on the angle γ also stabilize with time. Calculations by the loading scheme described agree well with results of laboratory experiments [8]. It should be noted that in the experiments of [8], about 8–10 rotations of the loading plates were required to stabilize the deformation process, i.e., $\gamma \sim 16\pi$–20π. For the model of [15], stabilization is attained much more rapidly, practically in one rotation ($\gamma \sim 2\pi$).

In describing complex loading, one of the main problem is to select the equation of coaxiality or disalignment for the principal axes of stresses. The experiments of [8] revealed that the principal axes of the stress tensor in the plane Ox_1x_2 coincide with the axes of the ellipse. The largest compressing stress is directed along the minor axis of the ellipse, and the smallest stress is directed along the major axis. In other words, in

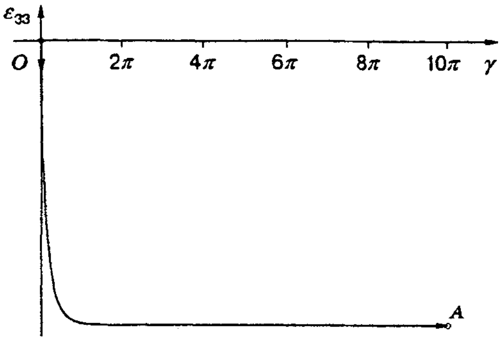

Fig. 4

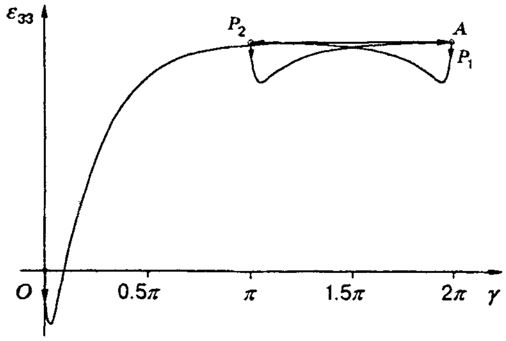

Fig. 5

a free-flowing medium, the condition of coaxiality of the stress and strain tensors is satisfied. In contrast, for viscous liquids, the stress-tensor is coaxial to the strain-rate tensor. Therefore, the principal axes of stresses are deflected from the major axis of the ellipse through an angle of $\pm 45°$ [8], which is confirmed by a direct experiment using the procedure described above. A numerical experiment for model (3)–(5) gives a deflection of the principal stress σ_1 from the major axis of the ellipse through an angle of $-43.85°$.

We now change the loading scheme and consider situations where the direction of rotation of the loading plates 5 (see Fig. 2) can change. We use all initial values of the parameters of the problem in the form (11) except for the porosity e. We set $e = 0.8$ and perform a numerical experiment by the following loading scheme. We first increase the angle γ from 0 to 2π and then change the direction of rotation (Ω on $-\Omega$) and decrease the angle γ from 2π to π. After that, we change the direction of rotation and perform loading from $\gamma = \pi$ to $\gamma = 2\pi$. Calculations show that a decrease in the initial porosity produces loosening of the material as a whole (Fig. 5; here and in Fig. 6 the loading curve is OP_1P_2A). A change in the direction of loading leads to sudden compaction of the material with subsequent loosening and stabilization. This result agrees well with the experiments of [8], from which it follows that with a cyclic change in the direction of loading at small amplitudes of the angle γ, it is possible to attain very high density of the material. An analysis of the stresses σ_{11}, σ_{22}, and σ_{12} shows that a change in the direction of loading leads to a sharp jump of stresses, after which, however, the deformation process is rapidly stabilized.

From the experiments of [8] it follows that the dilatancy depends on the magnitude of the additional load on the surface of the free-flowing medium. If the surface of the material is additionally loaded (the design of the device makes this possible), then, after attainment of a steady deformation regime, the packing of the particles will be denser than without additional surface loading. In numerical calculations, the additional loading can be modeled as follows. Let the initial stress σ_{33} be twice that in the previous calculations, i.e., $\sigma_{33} = 0.0075$ MPa, and the initial porosity $e = 0.8$. The remaining parameters are the same as in (11). Calculations show that the deformation process becomes steady at higher density of the medium (in Fig. 6, the porosity at the point A is equal to $e = 0.78365$) than in calculations without additional loading (in Fig. 5, the porosity at the point A is $e = 0.81678$).

Thus, the results of comparison of the numerical and laboratory experiments on complex loading of a free-flowing medium with continuous rotation of the principal axes of stresses lead to the following conclusion. Model (3)–(5) predicts the qualitative behavior of dilatancy and gives a satisfactory approximation for the level of the stress state of free-flowing media under complex loading. In the model proposed here and the model of a viscous liquid, the degree of coaxiality of the stress and strain tensors is identical.

Let us analyze results of calculations of complex loading with broken loading trajectory. We examine the following initial state of the medium:

$$\sigma_{33} = 0.0075 \text{ MPa}, \quad \sigma_{11} = \sigma_{22} = \xi\sigma_{33}, \quad \xi = 0.42, \quad e = 0.85.$$

As the loading parameter, we use the angle of displacement of the chamber γ (see Fig. 3). The loading program is as follows. Initially, we perform a cyclic shear of the material with amplitude $\gamma_{max} = 5°$, i.e., a unidirectional

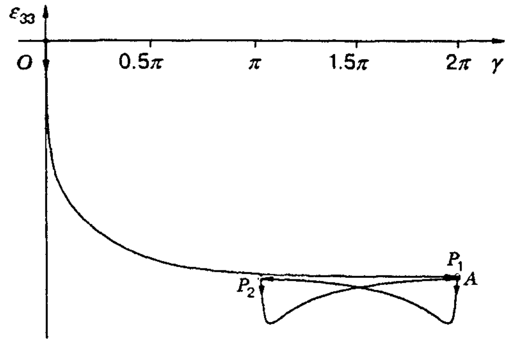

Fig. 6

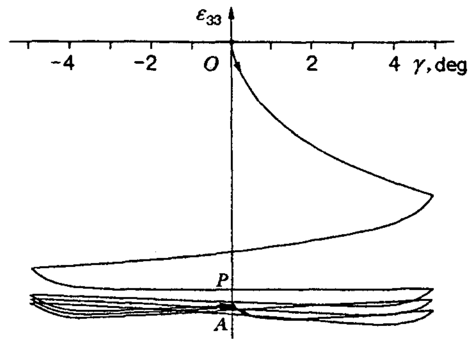

Fig. 7

shear (parameter $\delta = k_0 > 0$) to angle $\gamma = \gamma_{\max}$ and then a shear in the opposite direction ($\delta = -k_0 < 0$) to angle $\gamma = -\gamma_{\max}$, etc. In this case, the velocity field has the form (9). After a certain number of cycles, the trajectory is bent through angle æ at the moment $\gamma = 0$ (see Fig. 3b). The velocity field thus takes the form (10). After that, cyclic shear loading is continued by the scheme described above. Calculations were conducted for angles æ $= 30$, 60, and 90°.

The calculation results show that in cyclic shear until bending of the trajectory, the deformation process tends to a steady regime in which all parameters do not change from cycle to cycle but depend only on the phase (angle γ) inside the cycle. However, at the bend point of the trajectory, the steady regime is upset. Here sudden compaction of the material is observed. A plot of the strain ε_{33} versus γ is shown in Fig. 7 for æ $= 90°$. After the bending of the trajectory, a certain number of cycles is required again for stabilization of the deformation process. This result completely agrees with data of the laboratory experiments of [9].

Let us consider the problem of the coaxiality of the stress and strain tensors. In [5, 6], it is experimentally shown that in pure shear in a free-flowing medium, the condition of coaxiality of the principal axes of the stress and strain tensor is satisfied. As might be expected, bending of the loading trajectory leads to violation of this condition. Here disalignment of the directions of the principal stresses and strains by a particular angle is observed. We denote the disalignment angle by β. In the experiments, it was shown that at the point of bending of the trajectory through angles $10° <$ æ $< 80°$, the disalignment angle β depends weakly on æ and it is $\beta \approx 7.5°$. As æ $\to 0$ or æ $\to 90°$, the coaxiality is preserved since, in this case, the orientation of the principal axes of strains does not change and only the directions of minimum and maximal compression (in the case of æ $= 90°$) can vary. The subsequent cyclic shear loading leads to a gradual decrease in the angle β so that, as a result, the disalignment angle $\beta \to 0$. In other words, the bending of the loading trajectory was "erased" from the material's memory.

Calculations using model (3)–(5) are in qualitative agreement with results of these experiments. The calculated angle of disalignment β of the stress and strain tensors at the moment of bending of the trajectory was $\beta = 15.64°$ at æ $= 30°$, $\beta = 28.73°$ at æ $= 60°$, and $\beta = 0$ at æ $= 90°$. Subsequent shear deformation also results in the angle $\beta \to 0$.

Conclusions. The results obtained lead to the following conclusions.

The procedure and experiments of [5–9] on simple and complex loading of inelastic materials can be used as a basis for analysis of existing mathematical models of continuous media and development of new ones.

The hypoplastic model of a free-flowing medium of [15] gives a good qualitative approximation of real strain, predicts some properties of the material for different loading trajectories, and applies for boundary-value problems.

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